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Effects of a gravitational wave on relativistic particles†

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Abstract. The interaction of a gravitational wave with a beam of relativistic particles leads to effects which can be observationally significant. We find, in fact, that if a beam of monochromatic light is scattered by these particles, the frequency shift of the scattered photons, as seen by a stationary observer, undergoes fluctuations, due to the presence of the gravitational waves, which are larger than the amplitude of these by the particle Lorentz factor. The actual motion of the beam under the effect of a train of gravitational waves is discussed.

1. Introduction

The interactions of a gravitational wave (GW) with mechanical and electromagnetic systems have been investigated extensively with the aim of finding a proper detecting device. When acting upon a detector like a solid bar, a GW excites its quadrupole shear oscillations, inducing mechanical stresses (Lee *et al* 1976); acting on a static electromagnetic field, a GW produces electromagnetic waves (Boccaletti *et al* 1970, Papini and Valluri 1975). A suitable detection of these effects would provide evidence for the gravitational waves but unfortunately they are too weak to be experimentally significant. Gravitational waves also affect the motion of photons, so one would expect the luminosity of a distant source to suffer periodic fluctuations once the light propagates through them; here again there is little hope of discriminating between this effect and other competitive sources of fluctuations. Recently, considerable attention has been devoted to alternative and more sophisticated ways of detecting GW's using laser interferometry (Winkler 1976) and microwave cavity detectors (Caves 1978, Tourrenc 1978a, b, Pegoraro *et al* 1977, Braginsky *et al* 1977). In this paper I wish to consider the effects of a GW on test particles which are in relativistic motion with respect to stationary observers. The trajectory of a particle initially in motion in a direction transverse to the direction of propagation of a plane gravitational wave undergoes periodic changes not only transversely but also longitudinally to the GW. As a result, the particle will spiral around its main direction of motion as if it were moving through a magnetic field. Under these conditions let us assume that a beam of monochomatic light is scattered by these particles; the frequency shift of the scattered photons will fluctuate, as measured by a stationary observer, with an amplitude higher by a factor equal to the particle Lorentz factor than what it would be if the source and the observer

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were both stationary. A similar effect has been investigated by Mashhoon (1978), where the Doppler tracking of spacecraft was proposed as a possible tool to detect gravitational waves. These in fact could affect the motion of the source (a satellite) in such a way as to produce a secular change in the Doppler shift.

2. Frequency shift fluctuations

The metric of a weak, plane, elliptically polarised gravitational wave propagating in the x direction is conveniently written as

$$ds^2 = c^2\,dt^2 - (dx)^2 - (1-h_{22})\,dy^2 - (1+h_{22})\,dz^2 + 2h_{23}\,dy\,dz \tag{1}$$

where

$$h_{22} = h\sin[k(ct-x)+\psi] \qquad h_{23} = h'\sin[k(ct-x)+\psi']. \tag{2}$$

Here k is the wavevector; the amplitudes h and h' and the initial phases ψ and ψ' completely determine the state of polarisation of the GW. In terms of the retarded and advanced coordinates $u=\frac{1}{2}(ct-x)$ and $v=\frac{1}{2}(ct+x)$, it is easy to see that the coordinates y, z and v in the metric (1) may be ignored; therefore, calling $P_i=\mu c u_i$ the four-momentum of a free test particle, the three constants of the motion are defined by

$$P_y = -\mu c\alpha \qquad P_z = -\mu c\beta \qquad P_v = \mu c E. \tag{3}$$

Choosing the particle's proper time τ as the parameter on its trajectory, so that $u^i u_i = 1$, the components of the four-velocity are easily deduced and read

$$\begin{aligned} u^0 &= (1-f+E^2)/2E \qquad & u^y &= [\alpha(1+h_{22})+\beta h_{23}](1-h_{22}^2-h_{23}^2)^{-1} \\ u^x &= (1-f-E^2)/2E \qquad & u^z &= [\beta(1-h_{22})+\alpha h_{23}](1-h_{22}^2-h_{23}^2)^{-1} \end{aligned} \tag{4}$$

where

$$f = -(1-h_{22})(u^y)^2 - (1-h_{22})(u^z)^2 + 2h_{23}u^y u^z. \tag{5}$$

While the physical meaning of α and β is obvious, that of the constant E can be deduced from its very definition: it is the difference between the total specific energy of the particle and its specific momentum in the x direction. Its vanishing implies that the particle moves in the x direction with the velocity of light.

In the case of a null geodesic, one deduces similarly the components of its four-vector k^i as

$$\begin{aligned} k^0 &= (-\tilde f+\tilde E^2)/2\tilde E \qquad & k^y &= [\tilde\alpha(1+h_{22})+\tilde\beta h_{23}](1-h_{22}^2-h_{23}^2)^{-1} \\ k^x &= (\tilde E^2+\tilde f)/2\tilde E \qquad & k^z &= [\tilde\beta(1-h_{22})+\tilde\alpha h_{23}](1-h_{22}^2-h_{23}^2)^{-1} \end{aligned} \tag{6}$$

with

$$\tilde f = -(1-h_{22})k^y - (1+h_{22})k^z + 2h_{23}k^y k^z \tag{7}$$

and $\tilde\alpha$, $\tilde\beta$ and $\tilde E$ being the constants of the motion for the photon. As is well known, a GW induces a fluctuating frequency shift as a photon travels through. Let $u^i_{e(o)} = \delta^i_0$ be the unperturbed world lines of an emitter (observer) when no gravitational perturbation is present ($h_{ij}=0$). In this case the constants are simply $\alpha=\beta=0$, $E=1$; we then realise that these also remain stationary in the presence of a GW—the world lines $u^i = \delta^i_0$ are, in fact, geodesics for the metric (1). The proper spatial distance between the emitter and the observer, however, changes with time and this causes a frequency shift. To first order in h_{ij} and assuming for simplicity $\tilde{\beta} = 0$ (motion of the photon initially in the x, y plane), we have

$$1+Z=\frac{(u^ik_i)_\mathrm{e}}{(u^ik_i)_\mathrm{o}}\simeq\frac{1+(\tilde{\alpha}^2/\tilde{E}^2)[1+(h_{22})_\mathrm{e}]}{1+(\tilde{\alpha}/\tilde{E})^2[1+(h_{22})_\mathrm{o}]}\simeq 1+\frac{(h_{22})_\mathrm{e}-(h_{22})_\mathrm{o}}{1+(\tilde{E}/\tilde{\alpha})^2}. \tag{8}$$

We can express the ratio $(\tilde{\alpha}/\tilde{E})^2$ in terms of the angle between the directions of propagation of the photon and the GW; in our case we have

$$-k^x/k^0=\cos\phi\simeq 1-(\tilde{\alpha}/\tilde{E})^2(1+h_{22})+\mathrm{O}(h_{ij}), \tag{9}$$

which shows that the light beam fluctuates slightly around an average value of ϕ which we write as $2\sin^2\frac{1}{2}\phi_0=(\tilde{\alpha}/\tilde{E})^2$. The direction of the light beam, however, also fluctuates in the z direction. From (6), in fact, this is assured by the non-zero component of k^z which causes a fluctuation given by

$$-k^z/k^0=\cos\theta=2\frac{(\tilde{\alpha}/\tilde{E})h_{23}}{(1+\tilde{\alpha}^2/\tilde{E}^2)}=\frac{2\sqrt{2}\sin(\frac{1}{2}\phi_0)h_{23}}{(1+2\sin^2(\frac{1}{2}\phi_0))}. \tag{10}$$

Going back to (8) we have finally (Kaufmann 1970, Estabrook and Wahlquist 1975)

$$Z=\Delta\omega/\omega_0=[(h_{22})_\mathrm{e}-(h_{22})_\mathrm{o}]\frac{2\sin^2\frac{1}{2}\phi_0}{(1+2\sin^2\frac{1}{2}\phi_0)}+\mathrm{O}(h_{ij}). \tag{11}$$

From (9), (10) and (11) we find, as expected, two effects of a GW on a light beam, a change in the energy and a change in direction. The largest effect is clearly observed in the y direction ($\phi_0=\frac{1}{2}\pi$). In this case the image of the source will fluctuate around the y axis and will in general describe a distorted ellipse. The average amplitudes of these fluctuations are very small, being of the order of h_{ij}; however, as pointed out by Grishchuk (1974), the change in the energy as well as the displacement of the photon trajectory from the y axis can be systematically built up with a system of reflecting mirrors, suitably arranged. Let us instead suppose that the source is initially in motion along the y axis; its unperturbed trajectory is then described by the four-vector

$$u^i_{(\mathrm{e})}\equiv\{(1+\alpha^2)^{1/2},0,\alpha,0\} \tag{12}$$

with the obvious choice $\alpha\neq 0$, $\beta=0$, $E=(1+\alpha^2)^{1/2}$. In the presence of a GW, the motion of the source is perturbed and described by

$$\begin{aligned} u^0_{(\mathrm{e})}&=(1+\alpha^2)^{1/2}+\frac{\alpha^2h_{22}}{2(1+\alpha^2)^{1/2}} & u^y_{(\mathrm{e})}&=\alpha(1+h_{22})\\ u^x_{(\mathrm{e})}&=\frac{\alpha^2h_{22}}{2(1+\alpha^2)^{1/2}} & u^z_{(\mathrm{e})}&=\alpha h_{23} \end{aligned} \tag{13}$$

from (4) and (5).

The instantaneous spatial velocity of the particle source relative to a local stationary observer is given by

$$v^i=c(u^i-\delta^i_0u_0)/u_0 \tag{14}$$

and read from (13) to first order in h_{ij};

$$v^x = c[\alpha^2 h_{22}/2(1+\alpha^2)] \qquad v^y = c\frac{\alpha}{(1+\alpha^2)^{1/2}}\left(1+h_{22}\frac{2+\alpha^2}{2(1+\alpha^2)}\right)$$
$$v^z = c[\alpha h_{23}/(1+\alpha^2)^{1/2}] \qquad [1-(v^2/c^2)]^{-1/2} = \epsilon/\mu c^2 = (1+\alpha^2)^{1/2}\left(1+\frac{\alpha^2 h_{22}}{2(1+\alpha^2)}\right). \tag{15}$$

Here ϵ is the particle total energy; it is clear that a GW induces velocity fluctuations in the z as well as in the x direction, but these become significant (i.e. at least of the order of h_{ij}) only for particles with relativistic velocities ($\alpha \gg 1$). Let us now assume that these particles become sources of light signals; this can be obtained, for example, by letting them scatter with a light beam. The frequency of the scattered photons in a given direction is easily calculated and reads, with respect to a stationary observer ($u^i_{(o)} = \delta^i_0$) as

$$\begin{aligned} 1+Z &= \frac{1}{(k_0)_o}(u_i k^i)_e \\ &= \left(\frac{1}{k_0}\right)_o \left[\frac{1}{2\tilde{E}}(\tilde{E}^2 - \tilde{f})\left((1+\alpha^2)^{1/2} + \frac{\alpha^2 h_{22}}{2(1+\alpha^2)^{1/2}}\right)\right. \\ &\quad \left. + \frac{1}{2\tilde{E}}(\tilde{E}^2+\tilde{f})\frac{\alpha^2 h_{22}}{2(1+\alpha^2)^{1/2}} - \alpha[\tilde{\alpha}(1+h_{22}) + \tilde{\beta} h_{23}]\right]_{(e)}. \end{aligned} \tag{16}$$

Assuming as before $\tilde{\beta} = 0$, (16) becomes, to first order in h_{ij},

$$\begin{aligned} 1+Z &= \frac{1}{[1+(\tilde{\alpha}/\tilde{E})^2(1+h_{22})]_o}\left((1+\alpha^2)^{1/2}[1+(\tilde{\alpha}/\tilde{E})^2(1+h_{22})]\right. \\ &\quad \left. + \frac{\alpha^2 h_{22}}{(1+\alpha^2)^{1/2}} - 2\alpha(\tilde{\alpha}/\tilde{E})(1+h_{22})\right)_e. \end{aligned} \tag{17}$$

Here again the ratio $\xi = \tilde{\alpha}/\tilde{E}$ measures the average angle between the photon direction and the GW direction; it is also easy to show that in this case the largest effect is in the y direction ($\xi = \pm 1$). If the motion is relativistic, $\alpha \gg 1$, equation (17) becomes

$$1+Z = \frac{\alpha(1-\xi)^2}{(1+\xi^2)}\left(1+(h_{22})_e - \frac{\xi^2}{1+\xi^2}(h_{22})_o\right) + O(h). \tag{18}$$

We see now that, while a GW (1) had no effect on the photons emitted in the x direction, when the source is stationary here, due to the longitudinal (i.e. v^x) velocity perturbation induced by the GW, we have a nonzero transverse red-shift fluctuation given by

$$(1+Z) = \alpha + \alpha(h_{22})_e + O(h) \qquad \xi = 0. \tag{19}$$

In this case, as well as in the general case (18), the frequency shift varies with respto the average value given by the Doppler effect, with an amplitude of the order of $\alpha h \gg h$.

3. Trajectory fluctuations

Let us now consider the effect of the velocity perturbations on the motion of a beam of relativistic particles. Assuming $\alpha \gg 1$, equation (13) simplifies as

$$u^0 = \alpha + \tfrac{1}{2}\alpha h_{22} \qquad u^x = \tfrac{1}{2}\alpha h_{22} \qquad u^y = \alpha(1+h_{22}) \qquad u^z = \alpha h_{23}. \tag{20}$$

Let us further assume that the particles are emitted continuously at $x = y = z = 0$ along the y direction and hit a screen perpendicular to the y axis at a distance L from the origin. In the absence of GW's, the beam would produce, in the plane of the screen, a bright spot at $x = z = 0$. When a train of GW's acts upon the beam, then in general we expect the spot to describe a distorted ellipse. To see this, let us integrate equation (20); from the '0' and 'x' components we have

$$\mathrm{d}x^0 - \mathrm{d}x = 2\,\mathrm{d}u = \alpha\,\mathrm{d}\tau, \tag{21}$$

which implies $u = \frac{1}{2}\alpha\tau + A$, A being a constant. Now we choose A such that when $\tau = 0$, the ith particle starts at $x = y = z = 0$ and $t = t_i$, the latter being the initial coordinate time, so we have $A = \frac{1}{2}ct_i$. Recalling that $\mathrm{d}x = \mathrm{d}v - \mathrm{d}u$, we have from (20) and (21)

$$\mathrm{d}v = \mathrm{d}u + h\sin(2ku + \psi)\,\mathrm{d}u$$

which, upon integration, yields

$$x = -\frac{h}{2k}\cos[\alpha k\tau + \tfrac{1}{2}kct_i + \psi] + B. \tag{22}$$

Here B is a new constant of integration which we choose requiring that at $\tau = 0$, $x = 0$, so finally, repeating the argument for the other coordinates, we have

$$\begin{aligned} x &= -\frac{h}{2k}(\cos(\alpha k\tau + \tfrac{1}{2}kct_i + \psi) - \cos(\tfrac{1}{2}kct_i + \psi)) \\ y &= \alpha\tau - \frac{h}{k}(\cos(\alpha k\tau + \tfrac{1}{2}kct_i + \psi) - \cos(\tfrac{1}{2}kct_i + \psi)) \\ z &= -\frac{h'}{k}(\cos(\alpha k\tau + \tfrac{1}{2}kct_i + \psi') - \cos(\tfrac{1}{2}kct_i + \psi')). \end{aligned} \tag{23}$$

The correction to the y coordinate simply tells that the proper time spent by the particle to reach the screen at $y = L$ fluctuates around a mean value $\tau_s = \alpha L$. Neglecting, however, the y correction we have that, on the screen, the spot moves according to the law

$$\begin{aligned} x_s &= -\frac{h}{2k}(\cos(kL + \tfrac{1}{2}kct_i + \psi) - \cos(\tfrac{1}{2}kct_i + \psi)) \\ z_s &= -\frac{h'}{2k}\cos(kL + \tfrac{1}{2}kct_i + \psi') - \cos(\tfrac{1}{2}kct_i + \psi')). \end{aligned} \tag{24}$$

It is clear that the running coordinate here is the particle emission time: particles will hit the screen with different phases and therefore the spot will move on the screen. We can visualise this in the simple case of a GW polarised elliptically with $h = 2h'$ and $\psi' = \psi \pm \frac{1}{2}\pi$; in this case equations (24) become

$$\begin{aligned} x_s &= -\frac{h}{2k}(\cos(kL + \Phi_i) - \cos\Phi_i) \\ z_s &= \pm\frac{h}{2k}(\sin(kL + \Phi_i) - \sin\Phi_i) \end{aligned} \tag{25}$$

with $\Phi_i = \frac{1}{2}kct_i + \psi$. After simple algebra (25) becomes

$$x_s = +\frac{h}{k}\sin(\tfrac{1}{2}kL)\sin(\Phi_i + \tfrac{1}{2}kL)$$
$$z_s = \pm\frac{h}{k}\sin(\tfrac{1}{2}kL)\cos(\Phi_i + \tfrac{1}{2}kL). \tag{26}$$

Equation (26) is clearly the parametric equation of a circle with radius

$$R = h|\sin\tfrac{1}{2}kL|/k. \tag{27}$$

The largest effect is when the screen is at $L = \frac{1}{2}(2n+1)\lambda$ where λ is the GW wavelength, while it disappears when L is any integer multiple of λ. The overall motion of the particles in the beam is that of a circular helix in the y direction with a radius which fluctuates with twice the period of the GW. The amplitudes of these fluctuations, however, are negligibly small; on the Earth, typical values of h and λ due to cosmical explosions in our Galaxy are $h \sim 10^{-20}$ and $\lambda \sim 10^7$ cm so one would have from (27) a fluctuation of $R \sim h\lambda \sim 10^{-13}$.

4. Conclusions

The interaction between a train of plane gravitational waves and a beam of relativistic particles gives rise to effects which are worth mentioning. The particle trajectories in the beam undergo fluctuations around the main direction of the motion which, depending on the state of polarisation of the incident GW, results in a distorted helix.

If these particles are allowed to be sources of light signals, by a scattering process say, the frequency of the scattered photons, beside the expected Doppler shift, undergoes fluctuations which are of the order of the GW amplitude multiplied by the transverse momentum of the particles. Notably, the higher the energy of the source, the more significant are the corrections to the Doppler shift.

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